A physically based approach to the solution of the Fokker-Planck equation

O O Onyejekwe
National University of Science and Technology, Buluwayo, Zimbabwe

Abstract

Two motor forces, namely capillarity and gravity, play a major role as water moves down a soil profile. Initially capillary forces are dominant but assume a lesser role as infiltration becomes slowed down progressively by viscous forces. On the other hand, the gravity force is negligible at the beginning, but becomes quite active and noticeable rapidly. These changes suggest modification of solution methods for flow through porous media. In this study the non-linear Fokker-Planck equation is firstly non-dimensionalised to reflect the time-dependent motion of flow through a soil profile and the resulting equation is linearised with the Newton-Richtmeyer scheme. Some representative cases are studied with the numerical model and the results obtained are found to be physically realistic.

Introduction

Water in the soil is subjected to various factors, among which are diffusivity, evaporation, rate of water application and plant uptake. A combined effect of these factors determines a scalar profile, which can be quantified by numerical or analytical techniques. In the analytical approach, both the boundary conditions and the soil hydraulic properties are considerably simplified in order to enhance the development of quasi-analytical solutions. For example, the non-linear one-dimensional infiltration equation was solved by using the Boltzmann substitution to replace the two independent variables x and t (Philip, 1957). By integrating the new variable within the limits of initial soil moisture content, and soil moisture content at saturation, a new term, the 'sorptivity' was introduced. Similar solutions with implicit extraction functions have been recorded (Warrick, 1975; 1976).

One advantage of the analytical approach, despite the simplification of both the governing equations and their boundary conditions, is that the solutions so obtained, present in a more vivid way, the underlying physics of the flow process and their dependence on certain flow and soil physical parameters. In addition, they provide a means of checking numerical algorithms.

Numerical techniques, on the other hand, have the capability of handling more realistic boundary conditions without necessarily oversimplifying the governing equations. Finite difference solutions of unsaturated flow models can be found in the classical works of Hanks and Bowers (1962), and Klute et al. (1965). The application of finite element methods to subsurface flow is a fairly recent development. In this approach, the solution for any dependent variable is approximated by interpolating functions, which, when substituted in the original equation, results in a residual. The Galerkin method seeks to reduce this residual by integrating over the element area and equating to zero. By carrying out this process over the solution domain, a set of simultaneous equations is obtained which is solved to yield the scalar profile (Hayhoe, 1978; Neumann et al., 1975; Pinder and Frind, 1972).

Recently, a major interest in the solution of fluid movement in porous media is the consideration of the advancement of the solid-liquid interface. The key factor lies in the specification of the liquid position with respect to time. A similar approach had been adopted in heat transfer problems (Ockendon and Hodgkins, 1975; Gupta and Kumar, 1983; Wood, 1991).

The present study aims at producing a numerical model which reflects the physics of soil-water movement as well as the influence of different boundary conditions on the moisture content profile.

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Problem development

The evolution of water content in time and space as water moves down a soil profile can be described by the non-linear FokkerPlanck equation, namely:

$$\frac{\partial\theta}{\partial t} = \nabla . [D(\theta)\ \nabla\theta\ ] - \frac{\partial\ k(\theta)}{\partial\theta}\ \frac{\partial\theta}{\partial z} \qquad (1)$$

The first and second terms of the RHS respectively account for the effects of moisture gradients and gravity. If we consider both effects as equally important, the one-dimensional version of Eq. (1) is given by:

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\ [\ D(\theta)\ \frac{\partial\theta}{\partial z} - K(\theta)\ ] \qquad (2)$$

where:

t	=	time
θ	=	volumetric moisture content
$D(\theta)$	=	soil water diffusivity
$K(\theta)$	=	hydraulic conductivity
z	=	distance from the soil surface, positive downward.

Figure 1 illustrates a typical soil profile, with the z coordinate pointing positive downwards. Equation (2) is non-dimensionalised according to **Appendix 1** to yield:

$$\frac{\partial\theta}{\partial t^*} = \frac{\partial}{\partial z^*}\ [\ D(\theta)^*\ \frac{\partial\theta}{\partial z^*}] - \frac{\partial K^*}{\partial z^*} + \frac{z^*}{s}\ \frac{ds}{dt^*}\ \frac{\partial\theta}{\partial z^*} \qquad (3)$$

with:

$$-\ \frac{T}{s^2}\ D(\theta) \equiv D(\theta)^*\ ; \quad \text{and} \quad -\ \frac{T}{s}\ K(\theta) \equiv K^*$$

where :

$*$ = superscript to denote dimensionless quantities

T and s are characteristic time and distance parameters (see **Appendix 1**).

Equation (3) is then linearised with the Newton-Richtmeyer scheme (see **Appendix 1** for detail). The resulting equation is discretised to yield the delta formulation of a linear finite difference model, whose equation is described by :

$$\frac{\Delta\theta^{k+1}}{\Delta t} = \frac{\alpha^k_{i+1/2}(\theta^k_{i+1} - \theta^k_i) - \alpha^k_{i-1/2}(\theta^k_i - \theta^k_{i-1})}{(\Delta z^*)^2}$$

$$+ \frac{\alpha^k_{i+1/2}(\Delta\theta^{k+1}_{i+1} - \Delta\theta^{k+1}_i) - \alpha^k_{i-1/2}(\Delta\theta^{k+1}_i - \Delta\theta^{k+1}_{i-1})}{(\Delta z^*)^2}$$

$$+ \frac{\left(\frac{\partial\alpha}{\partial\theta}\right)^k_{i+1/2}(\theta^k_{i+1} - \theta^k_i)(\Delta\theta^{k+1}_{i+1} - \Delta\theta^{k+1}_i)}{2(\Delta z^*)^2}$$

$$- \frac{\left(\frac{\partial\alpha}{\partial\theta}\right)^k_{i-1/2}(\theta^k_i - \theta^k_{i-1})(\Delta\theta^{k+1}_i - \Delta\theta^{k+1}_{i-1})}{2(\Delta z^*)^2}$$

$$- \frac{K^k_{i+1} - K^k_{i-1}}{2(\Delta z^*)^2} \qquad (4)$$

where:

k is a superscript referring to time increment

ζ^k and α are as defined in **Appendix 1**

t is the time interval.

Equation (4) can be put in the tridiagonal form:

$$A^k_i \Delta\theta^{k+1}_{i+1} + B^k_i \Delta\theta^{k+1}_i + C^k_i \Delta\theta^{k+1}_{i-1} \qquad (5)$$

where A, B, C are coefficients of $\Delta\theta$. The Thomas algorithm or any appropriate method is employed to solve Eq. 4. To finally obtain the dependent variable, the delta formulation is resolved as shown:

$$\Delta\theta^{k+1}_i = \theta^{k+1}_i - \theta^k_i \qquad (6)$$

$$\theta^{k+1} = \theta^k_i + \Delta\theta^k_i \qquad (6a)$$

This calls for a complete updating of the tridiagonal matrix with time and position.

Numerical results

Two test cases serve to verify the accuracy and convergence of the solution scheme developed herein. The model is used to check the effect of horizontal movement of the wetting front on a barrier for the case of furrow irrigation leading to horizontal infiltration into a homogeneous soil (Braddock et al., 1982). Figure 2 shows 2 symmetrically placed ditches and a barrier at the right end. The physics of the one-dimensional horizontal movement of water in the system can be described by Eq. (2). To facilitate comparison,

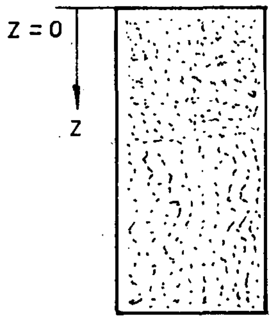

Figure 1
Soil profile

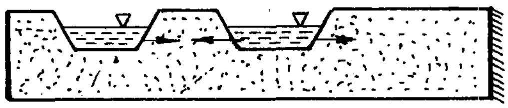

Figure 2
Furrow irrigation with a barrier

both the initial and boundary conditions,as well as the soil moisture diffusivity, are made the same as those specified by Braddock et al. (1982). Figure 3 shows that the front reaches the barrier at a time slightly less than 0,1 s. The front exhibits a sharp steepness up to 0,45 s because of the low value of the diffusivity. After this, the effect of the zero flux condition at the boundary becomes more noticeable as the water content between the boundary and the source tends to remain constant. There is a close agreement between the results obtained with this model and those of Braddock et al. (1982). Next, the model is applied to solve a non-linear PDE with a non-linear source term, and the steady state results obtained are compared with the analytical solution (see **Appendix 2** for the PDE, and analytical solution). Table 1 shows an excellent agreement between the numerical and analytical results.

TABLE 1
COMPARISON OF EXACT AND NUMERICAL RESULTS FOR SOIL MOISTURE CONTENT AT STEADY STATE

z	Exact	Numerical
0,0	1,0	1,0
0,1	0,9219	0,9220
0,2	0,8467	0,8469
0,3	0,7735	0,7734
0,4	0,7013	0,7020
0,5	0,6287	0,6290
0,6	0,5539	0,5541
0,7	0,4739	0,4736
0,8	0,3835	0,3836
0,9	0,2687	0,2700
1,0	0,0	0,0

One of the interesting aspects of this work is to observe the profiles of the scalar front within the problem domain. For this analysis, we considered a sinusoidal domain λ defined by :

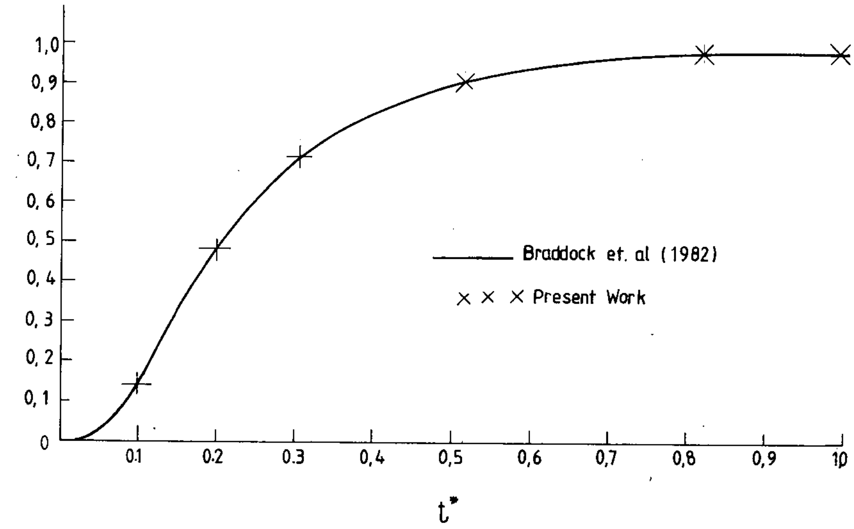

Figure 3
Profile of soil water concentration front at a barrier

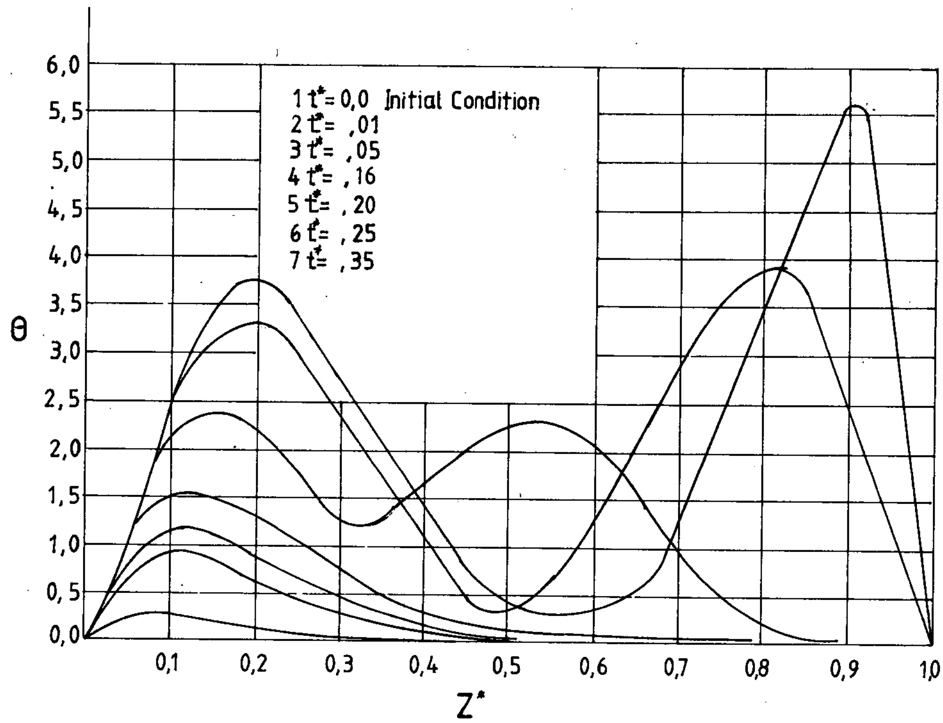

Figure 4
Profiles of scalar distribution in problem domain

$$\lambda = 0{,}5 \sin (2\pi t) + 1 \qquad (7)$$

The initial condition is defined by a polynomial:

$$\theta\ (t^* = 0) = 660 z^{*2}\ (1 - z^*)^9 + 2500 z^{*15} (1 - z^*)^2 \qquad (7a)$$

For the boundary conditions:

$$\theta = 0 \text{ at } z^* = 0, \text{ and } \theta = 0 \text{ at } z^* = 1 \qquad (7b)$$

both the conductivity and diffusivity are functions of moisture content and are respectively given by :

$$D(\theta) = \exp(1.32\theta) \qquad (7c)$$

$$K(\theta) = 4.86 * 10^{-5} \exp(2.56\theta) \qquad (7d)$$

Figure 4 shows the initial moisture distribution and the different profiles obtained at different time levels. At a small time interval, ($\Delta t = 0{,}01$), the profile still maintains its sinusoidal shape, despite the fact that a considerable movement, in a relative sense, has taken place from the lower boundary. At subsequent times the scalar profile is diffused and conducted within the soil profile; at the same time it tries to maintain the zero boundary conditions at the 2 ends. A steep scalar gradient is more noticeable at the beginning of the domain. The front is not well developed in this region, because of the relatively small value, and weak non-linearity of the diffusivity. As time increases, the influence of the boundary conditions on the dependent variable becomes more pronounced, the gradients decrease as both the conductivity and diffusivity assume constant values.

The model is now applied to study the movement in a soil characterised by the following hydraulic properties:

$$D(\theta) = \exp(1.5\theta) \quad (8)$$

$$K(\theta) = 8.56*10^{-8} \exp(3.46\theta) \quad (8a)$$

Most realistic infiltration problems involve alternate wetting, drying and moisture redistribution and hence require the imposition of Neuman or Cauchy type boundary conditions. For the case of continuous wetting, a Cauchy type boundary condition was imposed on the first computational node. This is defined by:

$$q_w^* = - D(\theta)* \frac{\partial\theta}{\partial z*} + K*(\theta) \quad (9)$$

Where q_w^* is the non-dimensional irrigation rate. For moisture redistribution :

$$0 = D*(\theta) \frac{\partial\theta}{\partial z*} + K*(\theta) \quad (9a)$$

Since this simulation involves a finite depth, the no-flux boundary condition is initially applied at the end region. This approach is valid before the approach of the wetting front. Thereafter it can be represented by a polynomial (Khan et al., 1982):

$$\theta_n = \theta(n-4) + 4(\theta(n-3) + \theta(n-1))-6(n-2) \quad (9b)$$

where n represents the last node in the problem domain. Two cases involving continuous wetting, intermittent wetting and soil moisture redistribution are tested. For continuous wetting application the soil is initially assumed wet and the boundary condition at the starting node is of the Cauchy type, with a 0,2 cm/d irrigation rate. A no-flux boundary condition is initially imposed at the terminal node. Figure 5 shows the profiles

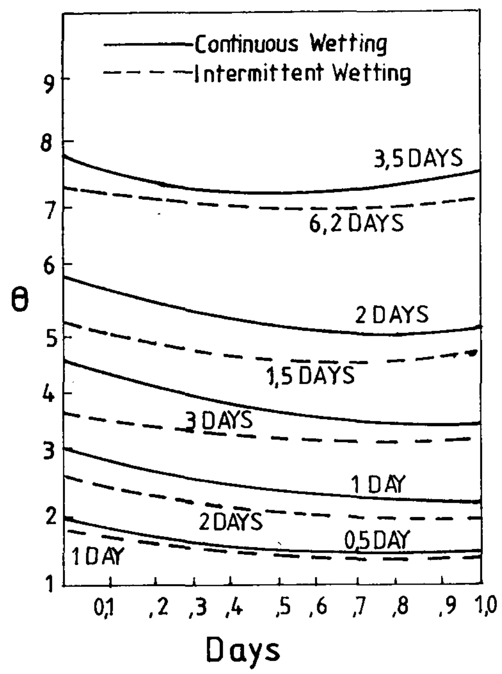

Figure 5

Soil moisture content distribution for continuous and intermittent wetting

obtained for different time levels. The scalar gradients decrease with time showing that the soil pores are filled continuously behind the wetting front. The flatter gradients correspond to when the soil is approaching saturation. At 3,5 d for example, the soil could be described as completely saturated. For the next case treated, instead of continuous wetting, equal times are allowed for intermittent wetting and soil moisture redistribution.

Equations (9) and (9a) are alternatively applied to the starting node of the problem domain. The downstream boundary is the same as that previously considered. Note that at each time level, less water is recorded at the starting node than in the previous case. More time is now allowed for the water to drain vertically. As the soil becomes progressively wet, the scalar gradient decreases. As expected, saturation is achieved at a relatively longer time, at 6,2 d.

Conclusions

This study has illustrated in a straightforward manner how to handle the one-dimensional version of the non-linear Fokker-Planck equation and adapt it to the practical aspects of the soil water movement in porous media. The response of the scalar field to realistic initial and boundary conditions has been studied. The results obtained show that the model is capable of explaining some pertinent aspects of water movement in porous media and is also capable of modelling various cases of infiltration with little adaptation.

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Appendix 1

$$\frac{\partial \theta}{\partial t} = \nabla.[D(\theta)\nabla \theta] - \frac{\partial k(\theta)}{\partial t}\frac{\partial \theta}{\partial z} \quad (1)$$

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}[D(\theta)\frac{\partial \theta}{\partial z} - K(\theta)] \quad (2)$$

If q = q(z*, t*) then:

$$\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial z^*}\frac{\partial z^*}{\partial t} + \frac{\partial \theta}{\partial t^*}\frac{\partial t^*}{\partial t} \quad (3)$$

Similarly:

$$\frac{\partial \theta}{\partial z} = \frac{\partial \theta}{\partial z^*}\frac{\partial z^*}{\partial z} + \frac{\partial \theta}{\partial t^*}\frac{\partial t^*}{\partial z} \quad (3a)$$

Definition:

$$z^* = \frac{z}{s(t)} \quad (3b)$$

From Eq. (3b):

$$\frac{\partial z^*}{\partial t} = \frac{z}{s^2}\frac{ds}{dt} \quad (3c)$$

Definition:

$$t^* = \frac{t}{T} \quad (3d)$$

Then:

$$\frac{\partial t^*}{\partial t} = \frac{1}{T} \quad (3e)$$

After substitution Eq. (3) can be simplified to give:

$$\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial z^*}(-\frac{z^*}{sT}\frac{ds}{dt}) + \frac{\partial \theta}{\partial t^*}(\frac{1}{T}) \quad (4a)$$

Similarly Eq. (3a) becomes:

$$\frac{\partial \theta}{\partial z} = \frac{\partial \theta}{\partial z^*}\frac{1}{s} \quad (4b)$$

Hence:

$$\frac{\partial k(\theta)}{\partial z} = \frac{\partial k(\theta)}{\partial z^*}\frac{1}{s} \quad (4c)$$

Following the method of Eq. (3a):

$$\frac{\partial}{\partial z}(\frac{\partial \theta}{\partial z}) = \frac{\partial z^*}{\partial z}\frac{\partial}{\partial z^*}(\frac{\partial \theta}{\partial z}) + \frac{\partial t^*}{\partial z}\frac{\partial}{\partial t^*}(\frac{\partial \theta}{\partial z}) \quad (4d)$$

But:

$$\frac{\partial \theta}{\partial z} = \frac{\partial \theta}{\partial z^*}\frac{1}{s} \;;\; \text{and} \; \frac{\partial t^*}{\partial z} = 0 \quad (4e)$$

$$\frac{\partial}{\partial z}(\frac{\partial \theta}{\partial z}) = \frac{1}{s}\frac{\partial}{\partial z}(\frac{\partial \theta}{\partial z^*}) \quad (4f)$$

According to Eq. (4d):

$$\frac{1}{s}\frac{\partial}{\partial z}\left(\frac{\partial\theta}{\partial z^*}\right) = \frac{1}{s}\frac{\partial z^*}{\partial z}\frac{\partial}{\partial z^*}\left(\frac{\partial\theta}{\partial z^*}\right) + \frac{\partial t^*}{\partial z}\frac{\partial}{\partial t^*}\left(\frac{\partial\theta}{\partial z^*}\right) \tag{4g}$$

It therefore folllows that:

$$\frac{\partial}{\partial z}\left(\frac{\partial\theta}{\partial z}\right) = \frac{1}{s^2}\frac{\partial^2\theta}{\partial z^{*2}} \tag{4h}$$

By the same token:

$$\frac{\partial}{\partial z}\left[D(\theta)\frac{\partial\theta}{\partial z}\right] = \frac{1}{s^2}\left[D(\theta)\frac{\partial^2\theta}{\partial z^{*2}}\right] \tag{4i}$$

substituting Eqs. (4), (4c) and (4i) into Eq. (2), we obtain:

$$\frac{\partial\theta}{\partial t^*}\left(\frac{-z^*}{sT}\frac{ds}{dt}\right) + \frac{\partial\theta}{\partial t^*}\left(\frac{1}{T}\right) = \frac{1}{s^2}\left[D(\theta)\frac{\partial^2\theta}{\partial z^{*2}}\right] - 1\frac{\partial k(\theta)}{\partial z^*} \tag{4j}$$

Hence:

$$\frac{\partial\theta}{\partial t^*} = \frac{\partial}{\partial z^*}\left[D(\theta)^*\frac{\partial\theta}{\partial z^*}\right] - \frac{\partial K^*}{\partial z^*} + \frac{z}{s}\frac{ds}{dt^*}\frac{\partial\theta}{\partial z^*} \tag{4k}$$

where:

$$\frac{T\,D(\theta)}{s^2} \equiv D(\theta)^* \quad \text{and ;} \quad -\frac{T}{s}k(\theta) \equiv K^*$$

For the linearisation method, apply the Newton-Richtmeyer scheme. Let $\alpha = D(\theta)^*$, then:

$$\left(\alpha\frac{\partial\theta}{\partial z^*}\right)_i^{k+1} = \left(\alpha\frac{\partial\theta}{\partial z^*}\right)_i^{k} + \Delta t\left(\frac{\partial}{\partial t}\left[\alpha\frac{\partial\theta}{\partial z^*}\right]\right)_i^{k} \tag{5}$$

$$\left(\alpha\frac{\partial\theta}{\partial z^*}\right)_i^{k+1} \cong \left(\alpha\frac{\partial\theta}{\partial z^*}\right)_i^{k} + \Delta t\left(\alpha\left(\frac{\partial^2\theta}{\partial t\partial z^*}\right) + \left(\frac{\partial\alpha}{\partial t}\right)\left(\frac{\partial\theta}{\partial z^*}\right)\right)_i^{k}$$

$$\cong \left(\alpha\frac{\partial\theta}{\partial z^*}\right)_i^{k} + \Delta t\left(\alpha\frac{\partial\theta}{\partial z^*}\left(\frac{\partial\theta}{\partial t}\right) + \left(\frac{\partial\alpha}{\partial\theta}\right)\left(\frac{\partial\theta}{\partial t}\right)\left(\frac{\partial\theta}{\partial z^*}\right)\right)_i^{k} \tag{5a}$$

Adopt a forward difference formula for the time term:

$$\left(\frac{\partial\theta}{\partial t}\right)_i^k = \frac{\theta^{k+1} - \theta^k}{\Delta t} = \frac{\Delta\theta}{\Delta t}{}_i^{k+1} \tag{5b}$$

$\Delta\theta^{k+1}$ now becomes the new independent variable. Eq. (5) now reads:

$$\left(\alpha\frac{\partial\theta}{\partial z}\right)_i^{k+1} = \left(\alpha\frac{\partial\theta}{\partial z}\right)_i^{k} + \alpha_i^k\frac{\partial\Delta\theta^{k+1}}{\partial z^*}{}_i + \left(\frac{\partial\alpha}{\partial z^*}\right)_i^k\left(\frac{\partial\theta}{\partial z^*}\right)_i^k \Delta\theta_i^{k+1} \tag{5c}$$

For the rest of the terms on the RHS of Eq. (4k) follows:

$$\left(\frac{\partial K^*}{\partial z}\right)_i^k = \frac{K^{*k}_{i+1} - K^{*k}_{i-1}}{2\Delta z^*} \tag{5d}$$

$$\left(\frac{z^*}{s}\frac{ds}{dt^*}\frac{\partial\theta}{\partial z^*}\right)_i^k = \zeta^k \tag{5e}$$

$$\zeta^k = \frac{i-1}{s}\frac{ds}{dt^*}\left(\frac{\theta_i^k - \theta_{i-1}^k}{\Delta z^*}\right) \quad \text{for } \frac{-z^*}{s}\frac{ds}{dt^*} > 0 \text{ (5} \tag{5f}$$

$$\zeta^k = \frac{i-1}{s}\frac{ds}{dt^*}\left(\frac{\theta_{i+1}^k - \theta_i^k}{\Delta z^*}\right) \quad \text{for } \frac{-z^*}{s}\frac{ds}{dt^*} < 0 \tag{5g}$$

We can now substitute Eqs. (5b), (5c), (5e) and (5f) into Eq. (4k) to give:

$$\left[\frac{\Delta\theta}{\Delta t*}\right]_i^{k+1} = \frac{\partial}{\partial z*}\left[\alpha\left(\frac{\partial\theta}{\partial z*}\right)\right]_i^k + \alpha_i^k \frac{\partial}{\partial z*}\Delta\theta_i^{k+1} + \left(\frac{\partial\alpha}{\partial\theta}\right)^k + \left(\frac{\partial\theta}{\partial z*}\right)_i^k \Delta\theta_i^{k+1} + \frac{K^{*k} - K_{i-1}^{*k}}{2\Delta z*} + \zeta^k \quad (5h)$$

(5h)

For ease of computation, the first term on the RHS of Eq. (5i) must be treated specially. The preferred way is given as follows:

$$\frac{\partial}{\partial z*}\left(\alpha\frac{\partial\theta}{\partial z*}\right)^k = \frac{\alpha_{i+1/2}^k\left(\frac{\partial\theta}{\partial z*}\right)_{i+1/2}^k - \alpha_{i-1/2}^k\left(\frac{\partial\theta}{\partial z*}\right)_{i-1/2}^k}{\Delta z*}$$

$$= \frac{\alpha_{i+1/2}^k(\theta_{i+1}^k - \theta_i^k) - \alpha_{i-1/2}^k(\theta_i^k - \theta_{i-1}^k)}{(\Delta z*)^2}$$

(5i)

Similarly:

$$\frac{\partial}{\partial z*}\left(\alpha_i^k\frac{\partial}{\partial z*}\Delta\theta_i^{k+1}\right) = \frac{\alpha_{i+1/2}^k(\Delta\theta_{i+1}^{k+1} - \Delta_i^{k+1})\ \alpha_{i-1/2}^k(\Delta\theta_i^{k+1} - \Delta\theta_{i-1}^{k+1})}{(\Delta z*)^2}$$

(5j)

Also:

$$\frac{\partial}{\partial z}\left[\left(\frac{\partial\alpha}{\partial\theta}\right)_i^k\left(\frac{\partial\theta}{\partial z*}\right)_i^k\Delta\theta_i^{k+1}\right] = \frac{\left[\left(\frac{\partial\alpha}{\partial\theta}\right)_{i+1/2}^k \cdot (\theta_{i+1}^k - \theta_i^k)\Delta\theta_{i+1/2}^{k+1} - \left(\frac{\partial\alpha}{\partial\theta}\right)_i^k(\theta_i^k - \theta_{i-1}^k)\Delta\theta_{i-1/2}^{k+1}\right]}{(\Delta z*)^2}$$

(5k)

The following approximations can be used for the delta terms:

$$\Delta\theta_{i+1/2}^{k+1} = \frac{\Delta\theta_{i+1}^{k+1} + \Delta\theta_i^{k+1}}{2} \quad (5l)$$

$$\Delta\theta_{i-1/2}^{k+1} = \frac{\Delta\theta_i^{k+1} + \Delta\theta_{i-1/2}^{k+1}}{2} \quad (5m)$$

Substituting Eqs. (5i), (5j), and (5k) into Eq. (5h) and factorising terms of the delta dependent variable yields:

$$-\Delta\theta_{i-1}^k[\alpha_{i-1/2}^k - (r/2)(\partial\alpha/\partial\theta)_{i-1/2}^k(\theta_i^k - \theta_{i-1}^k)]$$

$$= r\alpha_{i-1/2}^k(\theta_{i+1}^k - \theta_i^k) - \alpha_{i-1/2}^k(\theta_i^k - \theta_{i-1}^k)] +$$

$$(\Delta t*/2\Delta z*)[K_{i+1}^{*k} - K_{i-1}^{*k}] + \Delta t*\zeta^k \quad (5n)$$

where:

$$r = (\Delta t*/\Delta z*)$$

Appendix 2

Consider a non-linear PDE given by:

$$\partial\theta/\partial t = \partial/\partial z(\theta\ \partial\theta/\partial z) - \theta^2 \qquad (1)$$

I.C. : $\theta = 0$ at $t = 0$

B.C. : $\theta = 1$ at $z = 0$; and $\theta = 0$ at $z = 1$

At steady state, Eq. (1) is given by:

$$d/dz(\theta\ d\theta/dz) - \theta^2 = 0 \qquad (1a)$$

Multiply Eq. (1a) by 2 to give:

$$d/dz(2\theta\ d\theta/dz) - 2\theta^2 = 0 \qquad (1b)$$

Hence:

$$d^2\theta/dz^2 - 2\theta^2 = 0 \qquad (1c)$$

Let $\theta^2 = v$ (1d)

then Eq. (1c) becomes:

$$d^2v/dz^2 - 2v = 0 \qquad (1e)$$

Then the analytical solution of Eq. (1e) is given by:

$$v = A \sinh(z\surd 2) + B \sinh(z\surd 2) \qquad (1f)$$

At $z = 0$, $\theta = 1$; from equation (1d), $v = 1$

At $z = 1$, $\theta = 0$; from equation (1d), $v = 0$ (1g)

Substituting Eq. (1g) into Eq. (1f):

$$B = 1, \text{ and } A = -(\cosh\surd 2)/(\sinh\surd 2) = -\coth\surd 2 \qquad (1h)$$

Equation (1f) becomes:

$$v = [\ \cosh(z\surd 2) - \coth(z\surd 2)\ ] \qquad (1i)$$

Since $\theta^2 = v$

$$\theta = [\ \cosh(z\surd 2) - \coth(z\surd 2)\ ]^{.5} \qquad (1j)$$

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CONTENTS AND AUTHOR INDEX

Supplement to *Water SA* Vol. 20 No. 1

REFEREES

Herewith a list of referees who adjudicated the papers which appeared in the 4 issues published in 1993. We would like to thank them all most sincerely for their willingness and for the time and effort expended in reviewing these papers in the interest of *Water SA*. Without their valued input the journal cannot exist. (The number in brackets after the name indicates the number of papers which were adjudicated by a particular referee).

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CONTENTS

Vol 19 • No 1 • January 1993

Weak acid/bases and pH control in anaerobic systems — A review
RE Moosbrugger, MC Wentzel, GA Ekama and GvR Marais 1

Alkalinity measurement: Part 1 — A 4 pH point titration method to determine the carbonate weak acid/base in an aqueous carbonate solution
RE Moosbrugger, MC Wentzel, GA Ekama and GvR Marais 11

Alkalinity measurement: Part 2 — A 4 pH point titration method to determine the carbonate weak acid/base in aqueous solutions containing other weak acid/bases of known concentrations
RE Moosbrugger, MC Wentzel, GA Ekama and GvR Marais 23

Alkalinity measurement: Part 3 — A 5 pH point titration method to determine the carbonate and SCFA weak acid/bases in aqueous solution containing also known concentrations of other weak acid/bases
RE Moosbrugger, MC Wentzel, RE Loewenthal, GA Ekama and GvR Marais 29

Lauter tun (brewery) waste in UASB systems — Feasibility, alkalinity requirements and pH control
RE Moosbrugger, MC Wentzel, GA Ekama and GvR Marais 41

Grape wine distillery waste in UASB systems — Feasibility, alkalinity requirements and pH control
RE Moosbrugger, MC Wentzel, GA Ekama and GvR Marais 53

The selective cultivation of the thermotolerant *Aspergillus* sp. on spent sulphite liquor
WA Pretorius and GG Lempert 69

Growth characteristics of *Aspergillus* sp. grown on spent sulphite liquor
WA Pretorius and GG Lempert 73

Biomass production of *Aspergillus fumigatus* on spent sulphite liquor under non-aseptic conditions
WA Pretorius and GG Lempert 77

Three-dimensional simulation of glaciogenic seeding of clouds over Bethlehem
GW Reuter 81

Estimating areal mean monthly time series of rainfall
SD Lynch and MC Dent 85

A re-evaluation of the bilharzia risk in and around the Hartbeespoort Dam
KN de Kock, CT Wolmarans, S Nieuwoudt, MJ Smid and E Yssel 89

Supplement: Index to Volume 18

Vol 19 • No 2 • April 1993

Review of treatment performance at Hammarsdale Waste-water Works with special reference to alum dosing
DW de Haas, GP Borain and DA Kerdachi 93

Induction of nitrate build-up in water by some common disinfectants
SHJ Jooste and J van Leeuwen 107

Uranium sorption by glutamate glucan: A modified chitosan
Part I: Equilibrium studies
I Saucedo, E Guibal, J Roussy, Ch Roulph and P Le Cloirec 113

Uranium sorption by glutamate glucan: A modified chitosan
Part II: Kinetic studies
E Guibal, I Saucedo, J Roussy, Ch Roulph and P Le Cloirec 119

Membrane performance restoration. I: Abattoir process streams, cleaning regimes for UF membranes
EP Jacobs, P Swart, CJ Brouckaert and OO Hart 127

The Eastern Cape drought
MR Jury and K Levey 133

Determination of the profile available water capacity of maize and wheat at different growth stages
FMG Vanassche and MC Laker 139

Permeability prediction for water seepage through low porosity granular porous media
JP du Plessis and LI Roos 147

Investigations into the salinity preferences of successive larval developmental forms of five indigenous species of the freshwater prawn *Macrobrachium*
MC Cort and HJ Schoonbee 153

Occurrence of the red swamp crawfish *Procambarus clarkii* (Crustacea: Cambaridae) in the Crocodile River at Dullstroom, Transvaal
HJ Schoonbee 163

Technical note
Comparison of humidity chamber, MariSource hatching-tray and "Zuger" glass funnel incubation systems for breeding of *Cyprinus carpio* (L.) and *Clarias gariepinus* (Burchell)
JF Prinsloo, LC Hoffman and J Theron 167

Short communication
A method for discriminating between evaporation and seepage losses from open water canals
HH Bosman 171

Vol 19 • No 3 • July 1993

The physical and chemical limnology of Sterkfontein Dam, Eastern Orange Free State, South Africa
WG Dörgeloh, MT Seaman and IG Gaigher 177

Evaluation of the dual digestion system:
Part 1: Overview of the Milnerton experience
JR Messenger, HA de Villiers, SJA Laubscher, K Kenmuir and GA Ekama 185

Part 2: Operation and performance of the pure oxygen aerobic reactor
JR Messenger, HA de Villiers and GA Ekama 193

Part 3: Considerations in the process design of the aerobic reactor
JR Messenger and GA Ekama 201

Part 4: Simulation of the temperature profile in the batch fed aerobic reactor
JR Messenger and GA Ekama 209

Investigation of drinking- and Nile-water samples of upper Egypt
RM Awadallah, SS Ismael, MT Abd El Aal and ME Soltan 217

A note on the use of chlorine dioxide vs. chlorine for potable water treatment
R Marín Galvín and JM Rodríguez Mellado 231

Faecal coliform densities and water quality criteria in three coastal recreational lakes in the SW Cape, South Africa
WR Harding 235

A compatibility study of the effects of dairy and brewery effluents on the treatability of domestic sewage
JS Kilani 247

Effluent production and disposal in the South African dairy industry: A postal survey
JP Strydom, JF Mostert and TJ Britz 253

Adaptation of bacterial cultures to non-oxidising water treatment bactericides
VS Brözel, B Pietersen and TE Cloete 259

Development and verification of hydrograph routing in a daily simulation model
JC Smithers and RE Caldecott 263

Vol 19 • No 4 • October 1993

Effect of cadmium and copper on survival and reproduction of *Daphnia pulex*
DJ Roux, PL Kempster, E Truter and L van der Merwe 269

Towards integrated water quality monitoring: Assessment of ecosystem health
DJ Roux, HR van Vliet and M van Veelen 275

Bioaccumulation of iron in the freshwater crab (*Potamonautes warreni*) from three industrial, mine and sewage polluted freshwater ecosystems in the Transvaal
VE Steenkamp, HH du Preez, HJ Schoonbee, AJB Wiid and MM Bester 281

Determination and speciation of heavy metals in sediments of the Hartbeespoort Dam by sequential chemical extraction
PP Coetzee 291

The importance of weirs as refugia for hippopotami and crocodiles in the Limpopo River, South Africa
NHG Jacobsen and CJ Kleynhans 301

Die invloed van druppelgrootte, valhoogte en toedieningstempo op die generasie van afloopwater onder gesimuleerde oorhoofse besproeiing
AA Bloem en MC Laker 307

Laboratory-scale UASB digesters (with/without conditioning tank and recycle): Efficacy to treat increased hydraulic loads
BT Dudley, AR Howgrave-Graham, H Isherwood and E Senior 313

Influence of higher organic loading rates on the efficiency of an anaerobic hybrid digester while treating landfill leachate
C Myburg and TJ Britz 319

The effect of different urban development types on stormwater runoff quality: A comparison between two Johannesburg catchments
FR Wimberley and TJ Coleman 325

Hydrous zirconium (iv) ozide and zirconium polyelectrolyte membranes on porous stainless steel supports - The effect of modified substrate pore size on the flux and rejection characteristics of the membranes
FG Neytzell-de Wilde 331

The application of single-column ion chromatography for the determination of sulphate in rain-water samples
P Kyriacou and PL Kempster 339

Short communication
Application of Sterikon® bioindicators for the determination of bactericide concentrations
TE Cloete, E da Silva and VS Brözel 343

Errata 346

AUTHOR INDEX

Page numbers in **bold** refer to sole or first author:

Abd El Aal MT 217
Awadallah RM **217**
Bester MM 281
Bloem AA **307**
Borain GP 93
Bosman HH **171**
Britz TJ 253, 319
Brouckaert CJ 127
Brözel VS **259**, 343
Caldecott RE 263
Cloete TE 259, **343**
Coetzee PP **291**
Coleman TJ 325
Cort MC **153**
Da Silva E 343
De Kock KN **89**
De Villiers HA 185, 193
De Haas DW **93**
Dent MC 85
Dörgeloh WG **177**
Du Preez HH 281
Du Plessis JP **147**
Dudley BT **313**
Ekama GA 1, 11, 23, 29, 41, 53, 185, 193, 201, 209
Gaigher IG 177
Guibal E 113, **119**
Harding WR **235**
Hart OO 127
Hoffman LC 167
Howgrave-Graham AR 313
Isherwood H 313
Ismael SS 217
Jacobs EP **127**
Jacobsen NHG **301**
Jooste SHJ **107**
Jury MR **133**
Kempster PL 269, 339
Kenmuir K 185
Kerdachi DA 93
Kilani JS **247**
Kleynhans CJ 301
Kyriacou P **339**
Laker MC 139, 307
Laubscher SJA 185
Le Cloirec P 113, 119
Lempert, GG 69, 73, 77
Levey K 133
Loewenthal RE 29
Lynch SD **85**
Marais GvR 1, 11, 23, 29, 41, 53
Marín Galvín R **231**
Messenger JR **185, 193, 201, 209**
Moosbrugger RE **1, 11, 23, 29, 41, 53**
Mostert JF 253
Myburg C **319**
Neytzell-de Wilde FG **331**
Nieuwoudt S 89
Pietersen B 259
Pretorius, WA **69, 73, 77**
Prinsloo JF **167**
Reuter GW **81**
Rodríguez Mellado JM 231
Roos LI 147
Roulph CH 113, 119
Roussy J 113, 119
Roux DJ **269, 275**
Saucedo I **113**, 119
Schoonbee HJ 153, **163**, 281
Seaman MT 177
Senior E 313
Smid MJ 89
Smithers JC **263**
Soltan ME 217
Steenkamp VE **281**
Strydom JP **253**
Swart P 127
Theron J 167
Truter E 269
Van der Merwe L 269
Van Leeuwen J 107
Van Veelen M 275
Van Vliet HR 275
Vanassche FMG **139**
Wentzel MC 1, 11, 23, 29, 41, 53
Wiid AJB 281
Wimberley FR **325**
Wolmarans CT 89
Yssel E 89