A generalised groundwater flow equation using the concept of non-integer order derivatives

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Abstract

The classical Darcy law is generalised by regarding the water flow as a function of a non-integer order derivative of the piezometric head. This generalised law and the law of conservation of mass are then used to derive a new equation for groundwater flow. Numerical solutions of this equation for various fractional orders of the derivatives are compared with experimental data and the Barker generalised radial flow model for which a fractal dimension for the flow is assumed.

Keywords: porous media, Darcy Law, integro-differential equations

Introduction

A problem that arises naturally in groundwater investigations is to choose an appropriate geometry for the geological system in which the flow occurs. For example, one can use a model based on percolation theory to simulate the flow in a fractured rock system with a very large fracture density (Berkowitz and Balberg, 1993) or the parallel plate model (De Marsily, 1986) to simulate flow through a single fracture. However, there are many fractured rock aquifers where the flow of groundwater does not fit conventional geometries (Black et al., 1986). This is in particular the case with the Karoo aquifers in South Africa, characterised by the presence of a very few bedding parallel fractures that serve as the main conduits of water in the aquifers (Botha et al., 1998). Attempts to fit a conventional radial flow model to the observed drawdown, see (Van Tonder et al., 2001), always yield a fit that underestimates the observed drawdown at early times and overestimates it at later times, as illustrated in Fig. 1.

The deviation of observations from theoretically expected values is usually an indication that the theory is not implemented correctly, or does not fit the observations. To investigate the first possibility Botha (Botha et al., 1998) developed a three-dimensional model for the Karoo aquifer on the campus of the University of the Free State. This model is based on the conventional, saturated groundwater flow equation for density-independent flow:

$$S_0(\underline{x},t)\partial_t \Phi = \nabla \cdot [\underline{\underline{K}}(\underline{x},t)\nabla\Phi] + f(\underline{x},t) \qquad (1)$$

where:

- S_0 is the specific storativity
- $\underline{\underline{K}}$ the hydraulic conductivity tensor of the aquifer
- $\Phi(\underline{x},t)$ the piezometric head
- $f(\underline{x},t)$ the strength of any sources or sinks, with $\underline{x}$ and t the usual spatial and time coordinates
- ∇ the gradient operator
- ∂_t the time derivative

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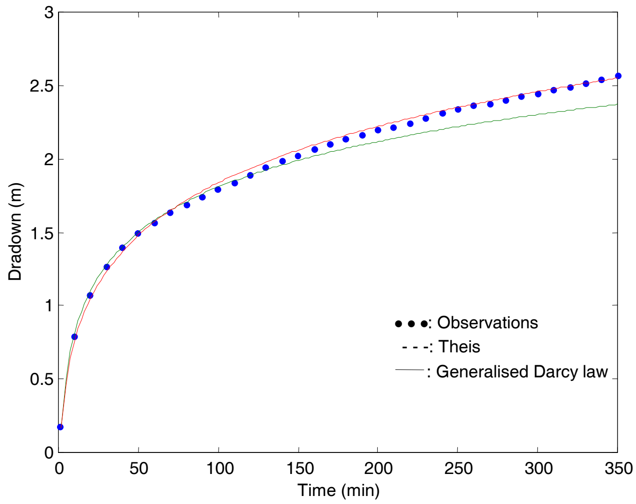

Figure 1
Comparison of the drawdown observed during a constant rate test on borehole UO5 at the Campus Test Site of the University of the Free State in South Africa with the fitted Theis curve and the solution of the generalised groundwater flow model with μ = 1.15; d = 40 mm.

This model showed that the dominant flow field in these aquifers is vertical and linear and not horizontal and radial as commonly assumed. However, more recent investigations (Van Tonder et al., 2001) suggest that the flow is also influenced by the geometry of the bedding parallel fractures, a feature that equation (1) cannot account for. It is therefore possible that the equation may not be applicable to flow in these fractured aquifers.

In an attempt to circumvent this problem, Barker (Barker, 1988) introduced a model in which the geometry of the aquifer is regarded as a fractal. Although this model has been applied with reasonable success in the analysis of hydraulic tests from boreholes in Karoo aquifers (Van der Voort, 2001), it introduces parameters for which no sound definition exist in the case of non-integer flow dimensions.

As a review of the derivation of Eq. (1) will show [see, e.g. (Bear, 1972)], Darcy law: