A physically based approach to the solution of the Fokker-Planck equation

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Abstract

Two motor forces, namely capillarity and gravity, play a major role as water moves down a soil profile. Initially capillary forces are dominant but assume a lesser role as infiltration becomes slowed down progressively by viscous forces. On the other hand, the gravity force is negligible at the beginning, but becomes quite active and noticeable rapidly. These changes suggest modification of solution methods for flow through porous media. In this study the non-linear Fokker-Planck equation is firstly non-dimensionalised to reflect the time-dependent motion of flow through a soil profile and the resulting equation is linearised with the Newton-Richtmeyer scheme. Some representative cases are studied with the numerical model and the results obtained are found to be physically realistic.

Introduction

Water in the soil is subjected to various factors, among which are diffusivity, evaporation, rate of water application and plant uptake. A combined effect of these factors determines a scalar profile, which can be quantified by numerical or analytical techniques. In the analytical approach, both the boundary conditions and the soil hydraulic properties are considerably simplified in order to enhance the development of quasi-analytical solutions. For example, the non-linear one-dimensional infiltration equation was solved by using the Boltzmann substitution to replace the two independent variables x and t (Philip, 1957). By integrating the new variable within the limits of initial soil moisture content, and soil moisture content at saturation, a new term, the 'sorptivity' was introduced. Similar solutions with implicit extraction functions have been recorded (Warrick, 1975; 1976).

One advantage of the analytical approach, despite the simplification of both the governing equations and their boundary conditions, is that the solutions so obtained, present in a more vivid way, the underlying physics of the flow process and their dependence on certain flow and soil physical parameters. In addition, they provide a means of checking numerical algorithms.

Numerical techniques, on the other hand, have the capability of handling more realistic boundary conditions without necessarily oversimplifying the governing equations. Finite difference solutions of unsaturated flow models can be found in the classical works of Hanks and Bowers (1962), and Klute et al. (1965). The application of finite element methods to subsurface flow is a fairly recent development. In this approach, the solution for any dependent variable is approximated by interpolating functions, which, when substituted in the original equation, results in a residual. The Galerkin method seeks to reduce this residual by integrating over the element area and equating to zero. By carrying out this process over the solution domain, a set of simultaneous equations is obtained which is solved to yield the scalar profile (Hayhoe, 1978; Neumann et al., 1975; Pinderand Frind, 1972).

Recently, a major interest in the solution of fluid movement in porous media is the consideration of the advancement of the solid-liquid interface. The key factor lies in the specification of the liquid position with respect to time. A similar approach had been adopted in heat transfer problems (Ockendon and Hodgkins, 1975; Gupta and Kumar, 1983; Wood, 1991).

The present study aims at producing a numerical model which reflects the physics of soil-water movement as well as the influence of different boundary conditions on the moisture content profile.

Problem development

The evolution of water content in time and space as water moves down a soil profile can be described by the non-linear FokkerPlanck equation, namely:

$$\frac{ae}{at} = v.\ [D(e)\ ve\ ] - \frac{a\ k(e)}{a0}\ \frac{\partial\theta}{az} \qquad (1)$$

The first and second terms of the RHS respectively account for the effects of moisture gradients and gravity. If we consider both effects as equally important, the one-dimensional version of Eq. (1) is given by:

$$\frac{ae}{at} = \frac{a}{dz}\left[\ D(e)\ \frac{ae}{az} - K(8)\ \right] \qquad (2)$$

where:

t	=	time
6	=	volumetric moisture content
D(9)	=	soil water diffusivity
K(8)	=	hydraulic conductivity
z	=	distance from the soil surface, positive downward.

Figure 1 illustrates a typical soil profile, with the z coordinate pointing positive downwards. Equation (2) is non-dimensionalised according to **Appendix 1** to yield:

$$\frac{\partial\theta}{\partial t*} = \frac{\partial}{\partial z*}\left[\ D(\theta)*\ \frac{\partial\theta}{\partial z*}\right] - \frac{\partial K*}{\partial z*} + \frac{z*}{s}\ \frac{ds}{dt*}\ \frac{\partial\theta}{\partial z*} \qquad (3)$$

with:

$$-\ \frac{T}{s^2}\ D(\theta) = D(\theta)*\ ; \quad \text{and} \quad -\ \frac{T}{s}\ K(\theta) = K*$$

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